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DISCRETE GENERALIZED GEOMETRIC DISTRIBUTIONS
FOR A CLASS OF NETWORKS OF QUEUES

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#431

Abstract:

Three new iterative calculations which converge to the ratio matrix of the discrete generalized geometric distribution are presented for queueing systems having this limiting distribution. Two criteria for convergence are provided.

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MEMBERS OF THE
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IN
REPLY
TO
A
LETTER
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In many congestion systems a single queue of customers is processed by a complex service system. Because only some components of the service system are in use and/or a finite number of jobs are partially processed, the service system can be in any one of m states ($m < \infty$). The natural stochastic model uses a two dimensional state of the system variable $N(t) = (N_1(t), N_2(t))$ with N_1 measuring the number of jobs in the system and N_2 identifying which of the m possibilities is the current condition of the service system. Although the process can be considered in continuous time and made Markovian by adding an appropriate number of continuous supplementary variables [3,4], here only an approximating discrete time discrete state chain will be considered. This assumption simplifies both notation and analysis but probably is immaterial for modeling uses especially if discrete supplementary variables are used when appropriate. It is further assumed that jobs are processed individually with the time interval chosen small enough so that at most, one event, job arrival, job departure, or change of processing condition occurs with a significant probability in a single interval. This means that the transition operator of the process is non zero only for states differing by at most 1 in N_1 . Finally it is assumed that for all values of N_1 from some point on the transition structure is constant in this variable. Qualitatively, this is equivalent to allowing various adjustments to the load when few customers are in the system, but once the system is sufficiently congested no further adaptation is possible. Letting $P_i(t)$ be the vector of probabilities of the m states for which $N_1 = i$ for i sufficiently large ($i \geq i^*$) the chain satisfies

$$P_i(t+1) = P_{i-1}(t)U + P_i(t)(I+B) + P_{i+1}(t)D$$

for matrices of probabilities U , $I+B$, and D (I is the identity matrix). It is assumed that all states communicate. Moreover, $U + I + B + D$ is the matrix of an m state irreducible acyclic chain. Finally, there is a k such that the sum of all k term products of the matrices U , $I + B$, and D such that the numbers of D 's is the same as the number of U 's and the cumulative numbers of D 's in the partial products is the same or less than the number of U 's, is a strictly positive matrix. This means there is some sequence of transitions from any state for which $N_1 = i$ to another state for which $N_1 = i$ though states for which $N_1 \geq i$. In [4] it was shown that, if they exist, the limiting probabilities for this system are geometric in the upper tail. Let $P_i = \lim P_i(t)$. For $i \geq i^*$

$$P_i = P_{i-1}UZ$$

where Z is a matrix of the expected number of visits starting from a state in set i to other states in set i through states with $N_1 \geq i$. The matrix Z is strictly positive (>0) and can be computed iteratively from

$$(1) \quad \begin{aligned} Z_1 &= -B^{-1} \\ Z_n &= -B^{-1}(I + UZ_{n-1}DZ_{n-1}) \end{aligned}$$

and satisfies

$$(2) \quad I + BZ + UZDZ = 0$$

Even in the previous discussion possible simplification of the calculation because of degeneracy was considered. In earlier work [1] an alternative iteration was developed under the special assumption $U = \lambda I$ for a number λ . In both cases, non degenerate limiting probabilities were assumed to exist. A condition for the existence of limiting probabilities in the special case was proved [2]. This paper extends this condition to the more general situation and relates this condition to the limiting probabilities. In addition, alternative iterative calculations for the matrix are established.

Q Iteration

Define the two iterative schemes

$$\begin{aligned}
 (3) \quad Q_0 &= WP \\
 Q_n &= (U + Q_{n-1}^2 D) (-B^{-1}) \\
 Q_0^* &= 0 \\
 (4) \quad Q_n^* &= (U + Q_{n-1}^{*2} D) (-B^{-1})
 \end{aligned}$$

The row vector P is strictly positive, W is a non negative column vector, and the scalar product $(P, W) = \Gamma > 0$. The vectors P and W satisfy

$$\begin{aligned}
 (5) \quad P(U + \Gamma B + \Gamma^2 D) &= 0 \\
 (6) \quad (U + \Gamma B + \Gamma^2 D)W &= 0
 \end{aligned}$$

The number Γ is the smallest positive number for which there is a $P \gg 0$ satisfying (5). Both schemes converge to the same limit. The limit has a

strictly positive i th row if there is a positive entry in the i th row of U .
The limit has eigenvector P and eigenvalue Γ .

The communication assumption implies that $I + B$ is not conservative and thus $-B^{-1} = \sum (I+B)^k$ exists. The non negativity of U , D and $-B^{-1}$ means that by induction $Q_n^* \geq Q_{n-1}^*$ and $Q_n \geq Q_n^*$. In addition an induction starting from $PQ_0 = PWP = \Gamma P$ proves $PQ_n = \Gamma P$. This follows because

$$\begin{aligned} PQ_n &= P(U + Q_{n-1}^2 D)(-B^{-1}) \\ &= P(U + \Gamma^2 D)(-B^{-1}) \end{aligned}$$

Using (5)

$$PQ_n = P(-\Gamma B)(-B^{-1}) = \Gamma P$$

Since P is strictly positive this implies that the coordinates P_r and P_s satisfy

$$P_r Q_{n,r,s} \leq \sum_k P_k Q_{n,k,s} = \Gamma P_s$$

From this the matrix entries $Q_{n,r,s}$ satisfy

$$Q_{n,r,s} \leq \Gamma \max_{rs} P_s / P_r < \infty$$

Since $Q_n \geq 0$, $Q_{n,r,s}$ is bounded and therefore $Q_{n,r,s}^*$ is bounded. This and the monotonicity of Q_n^* imply that the sequence Q_n^* converges to a finite limit Q^* .

Notice that Q_1^* includes all products of matrices of the form $U(I+B)^k$. Q_2^* includes the additional products of terms of the form $U(I+B)^{k_1} U(I+B)^{k_2} D(I+B)^{k_3}$. Generally Q_n^* includes all products of terms of the form $U(I+B)^{k_1}$

$U(I+B)^{k_2} D(I+B)^{k_3} U(I+B)^{k_4} \dots$ which include at most $n-1$ D's. All products begin with a U and the number of D's is one less than the number of U's. Rewriting Q_n^* as $U Z_n^*$, Z_n^* includes all n term products of the matrices $U(I+B)$ and D in which the products have the same number of U's and D's and the partial products all have the numbers of U's equal to or greater than the number of D's. The communication assumption implies that for n sufficiently large, $Q_n^* = U Z_n^*$ where Z_n^* is strictly positive. Thus Q_n^* has a strictly positive i th row if there is a non zero entry in the i th row of U and an i th row of 0's if U has no positive entry in the i th row. By renumbering states if necessary

$$Q_n^* = \begin{pmatrix} X_n & Y_n \\ 0 & \end{pmatrix}$$

where $X_n \gg 0$ and square, $Y_n \gg 0$. Now positive operator theory [5,6] implies that X_n has a positive eigenvalue α_n and a positive eigenvector $P_{n,1}^*$. Define $P_{n,2}^* = 1/\alpha_n P_{n,1}^* Y_n$. $P_{n,2}^* \gg 0$ and the vector $P_n^* = P_{n,1}^*, P_{n,2}^*$ is a positive eigenvector for Q_n^* for eigenvalue α_n . Let the limit of Q_n^* as n goes to infinity be

$$Q^* = \begin{pmatrix} X & Y \\ 0 & \end{pmatrix}$$

with P^* and α the positive eigenvector and eigenvalue. Again from positive operator theory the powers $1/\alpha Q^*$

$$(1/\alpha Q^*)^j = \begin{pmatrix} (1/\alpha X)^j & (1/\alpha X)^{j-1} (1/\alpha Y) \\ 0 & \end{pmatrix}$$

The powers $(1/\alpha X)^j$ converge to a matrix of the form $W P_1^*$ where $XW = \alpha W$. Thus $(1/\alpha X)^{j-1} (1/\alpha Y)$ converges to $W(P_1^* (1/\alpha Y)) = W P_2^*$ and $(1/\alpha Q^*)^j$ converges $W P^*$.

Since

$$Q = (U + Q^{*2}D)(-B^{-1})$$

$$0 = (1/\alpha Q^*)^j U + \alpha(1/\alpha Q^*)^{j+1} B + \alpha^2(1/\alpha Q^*)^{j+2} D$$

In the limit as j goes to infinity the powers converge to the common limit.

This means that the vector $P^{**} > 0$ and $\alpha > 0$ satisfy

$$(7) \quad P^*(U + \alpha B + \alpha^2 D) = 0$$

Moreover, an elementary positive operator result guarantees that since

$Q_n > Q_n^*$ and Q_n has eigenvalue Γ , the eigenvalue $\alpha_n \leq \Gamma$. Thus $\alpha \leq \Gamma$. By

assumption Γ is the smallest number for which there was a strictly positive vector P satisfying this equation (5). Thus $\alpha = \Gamma$. Since $Q_n \geq Q_n^*$, $P^* Q_n \geq P^* Q_n^*$. For large n this implies $P^* Q_n \geq \Gamma P^*$.

To continue, it is necessary to show that this inequality is an equality.

First, since $P > 0$ and $PQ_n = \Gamma P$, Γ is the largest eigenvalue of Q_n in

absolute value. For any non-negative matrix the eigenvalue of largest abso-

lute value is some non negative real number and has a non-negative eigen

vector. Let these be Γ and W for Q_n as an operation on the dual space.

In order that

$$\Gamma(P, W) = (PQ_n, W) = (P, Q_n W) = r(P, W)$$

it is necessary that $r = \Gamma$ since $P > 0$ and $W \geq 0$, $W \neq 0$ imply $(P, W) > 0$. Since

Γ is the largest eigenvalue, if Q_n is irreducible, the inequality

$P^* Q_n \geq \Gamma P^*$ must be an equality [5]. If Q_n is reducible, examine the irreducible matrix

$$Q_n + \epsilon(\epsilon/|P^*|)LP^*$$

where L is the column vector of 1's and $| \cdot |$ is the sum of the absolute values of the coordinates. Moreover this positive matrix cannot have a largest eigenvalue which exceeds the sum of the two largest eigenvalues $\Gamma + \epsilon$. Since

$$P^*(Q_n + \epsilon/|P^*|LP^*) \geq \Gamma P^* + \epsilon P^* = (\Gamma + \epsilon)P^*$$

the vector P^* must be an eigenvector for eigenvalue $\Gamma + \epsilon$ for the perturbed matrix. Thus

$$P^*(Q_n + \epsilon/|P^*|LP^*) = (\Gamma + \epsilon)P^*$$

Since this is true for all $\epsilon > 0$, letting ϵ go to 0 proves

$$P^* Q_n = \Gamma P^*$$

Since P^* must be an eigenvector of Q_n for eigen value Γ , the limit of $P^*(Q_n - Q_n^*) = 0$. Since $Q_n - Q_n^*$ must be non-negative and P^* is strictly positive, $P^*(Q_n - Q_n^*)$ approaches 0 implies that Q_n approaches Q_n^* . Thus Q_n has the same limit Q^* as Q_n^* and $P^* = P$.

An interesting consequence of this result is that the iterative scheme will converge for any initial matrix Q_0' which is non negative and for which $PQ_0' \leq \Gamma P$. For such a matrix

$$Q'_n \geq 0$$

$$Q'_n \geq Q_n^*$$

and

$$\Gamma P \geq PQ'_n$$

In the limit $\Gamma P \geq PQ'_n$ and $Q'_n \geq Q_n^*$ produce a contradiction unless limit $PQ'_n = \Gamma P$ and then limit Q'_n must be limit Q_n^* .

Another similar consequence is that if Q' is any non negative matrix which satisfies

$$(8) \quad Q' = (U + Q'^2 D)(-B^{-1})$$

with largest eigenvalue $\alpha \leq \Gamma$ then $Q' = \lim Q_n = \lim Q_n^*$. The iteration starting from $Q_0 = Q'$ merely reproduces Q' and $Q' \geq Q_n^*$. In the limit $P^* Q' \geq P^* Q_n^* = \Gamma P^*$ shows that $\alpha \geq \Gamma$. Thus $\alpha = \Gamma$. The combination of $Q' \geq Q^*$ and $P^*(Q' - Q^*) = 0$ imply that $Q' = Q^*$. Thus there is a unique non negative Q with largest eigenvalue $\leq \Gamma$ satisfying (8).

Another important property of this process is that $QDL = UL$ where L is the vector of 1's. Since $Q_0^* = 0$, $Q_0^* DL = 0 \leq UL$. In general

$$-UL = Q_n^* BL + Q_{n-1}^{*2} DL$$

$$\text{Since } Q_n^* \geq Q_{n-1}^* \geq 0$$

$$-UL \leq Q_n^* (BL + Q_{n-1}^* DL)$$

Since $BL = -UL - DL$, and the induction hypothesis that $Q_{n-1}^* DL \leq UL$ imply

$$-UL \leq Q_n^*(-UL-DL+UL)$$

or

$$UL \geq Q_n^*DL$$

If in the limit $UL \geq QDL$ with strict inequality in some coordinate then since $P \gg 0$

$$PUL > PQDL = \Gamma PDL$$

This contradicts

$$0 = P(U + \Gamma B + \Gamma^2 D)L = (1 - \Gamma)P(UL - \Gamma DL)$$

from which $PUL = \Gamma PDL$. Thus $QDL = UL$.

This iterative scheme produces the same limit as iteration (1). In previous work it was shown that iteration 1 converges for any Z_0 for which $Z_0 \geq -B^{-1}$ and $ZDL = L$. From the current iteration define Z' by

$$Z' = \sum_{n=0}^{\infty} (I + B + QD)^n = -(B + QD)^{-1}$$

Since $QDL = UL$, the series converges. Moreover $UZ' = -U(B + QD)^{-1} = Q$ and $Z' \geq -B^{-1}$. Finally

$$-L = Z'(B + QD)L = Z'(-DL)$$

and $Z'DL = L$. Thus if Z' is used as the initial condition for (1) the result is Z . Convergence is immediate since

$$Z_1' = -B^{-1} + (-B^{-1})QDZ' = -B + (-B^{-1})(-I - BZ') = Z'$$

Thus the two processes produce identical results.

Alternative Iteration

There is yet another iterative process which can be used to compute the ratio matrix. As in the special case in which U is a constant times the identity [1], this process uses an inversion. The iteration is

$$(9) \quad \begin{aligned} Z_0 &= 0 \\ Z_n &= -(B + UZ_{n-1}D)^{-1} \end{aligned}$$

Starting from the obvious $Z_1 = -B^{-1} \geq Z_0$, an induction proves $Z_n \geq Z_{n-1} \geq 0$. In addition, $Z_0 DL = 0 \leq L$, $Z_n DL \leq L$ is needed to show that the inverse exists at each stage. Combined with the communication assumption this guarantees that $I + B + UZ_n D \geq 0$ and $(I + B + UZ_n D)L \leq L - DL \leq L$. The inequality must be strict in at least one coordinate, since $D \neq 0$. Thus $I + B + UZ_n D$ is the transition matrix of a finite state Markov chain which is not conservative. Thus the matrix powers converge to 0 so that

$$Z_{n+1} = \sum_{n=0}^{\infty} (I + B + UZ_n D)^n = -(B + UZ_n D)^{-1}$$

exists. Since $Z_n \geq Z_{n-1} \geq 0$, $U \geq 0$, $D \geq 0$

$$UZ_{n-1}D \leq UZ_n D$$

$$(I + B + UZ_{n-1}D)^n \leq (I + B + UZ_n D)^n \text{ for all } n$$

and $Z_{n+1} \geq Z_n$. Since $D \geq 0$, $Z_{n+1}D \geq Z_n D$ and $0 \leq Z_n DL \leq L$, the entries in $Z_n D$ form monotonic nondecreasing sequences bounded by 1. Thus $Z_n D$ converges, and this implies that $B + UZ_n D$ or $-Z_{n+1}^{-1}$ converge. This in turn implies the convergence of Z_n to a matrix Z .

An obvious question is whether this process converges to the limit of the previous process. Using P, Γ, Q from the previous section define Z_n^* by

$$-Z_1^{*-1} = B + QD$$

$$-Z_{n+1}^{*-1} = B + UZ_n^*D$$

First $P(-Z_1^{*-1}) = P(B+QD) = P(B+\Gamma D) = -P(1/\Gamma U)$. Since $QDL = UL$, $(-Z_1^{*-1})^{-1}$ exists and $\Gamma P = PUZ_1^*$. Assuming $UZ_n^*DL = UL$, $-Z_{n+1}^{*-1}L = -DL$ and $(I+B+UZ_n^*D)L \leq L$ with strict inequality in at least one coordinate. Thus

$$\sum_{n=0}^{\infty} (I+B+UZ_n^*D)^n = -(B+UZ_n^*D)^{-1} = Z_{n+1}^*$$

exists. Moreover $L = Z_{n+1}^*DL$ and $UL = UZ_{n+1}^*DL$. In addition assume $\Gamma P = PUZ_n^*$

$$P(-Z_{n+1}^{*-1}) = P(B+UZ_n^*D) = P(B+\Gamma D) = -1/\Gamma PU$$

and

$$\Gamma P = PUZ_{n+1}^*$$

Finally $-Z_1^{*-1} \geq -Z_1^{-1}$ or $Z_1^* \geq Z_1$ and inductively $Z_n^* \geq Z_n$ implies $-Z_{n+1}^{*-1} \geq$

$-Z_{n+1}^{-1}$ or $Z_{n+1}^* \geq Z_{n+1}$. Thus $PUZ_n \leq PUZ_n^* = \Gamma P$. The limit does not reverse the inequality. Thus by the last section, the process

$$Q_0^{**} = UZ$$

$$Q_n^{**} = (U+Q_n^{**2}D)(-B^{-1})$$

must converge to Q . The convergence is immediate since the limit Z satisfies

$$UZUZD = -U-UZB$$

or,

$$Q_1^{**} = -(U+UZUZD)(-B^{-1}) = (-UZB)(-B^{-1}) = UZ$$

Thus $Q = UZ$.

The difference between the iteration schemes is the rate at which they collect the required products of powers of U , $I + B$ and D . The alternative process increases the number of these more rapidly but at a cost of an inversion. Moreover $Z \gg 0$ but Q has rows of 0's in rows in which U has rows of 0's. Thus the previous iteration can more effectively use special rapid matrix multiplication to reduce calculation time. Which should be used probably depends on the specific forms of U , B and D .

Although the monotonicity is lost there may be an advantage to starting this alternative iteration with any $Z_0^{**} \geq 0$ for which $Z_0^{**}DL \approx L$. In particular such a matrix can be formed as $-B^{*-1}$ where $B^* = B$ plus a matrix which is 0 except for diagonal entries which are the coordinates of UL . Probabilistically this modification is to ignore possible moves to the higher states. It makes the iterative process for $M/M/1$ converge in one step. Unfortunately this improvement is the exception not the rule.

Starting from Z_0^{**} inductively $-Z_n^{**1} \geq -Z_n^{-1}$ and $Z_n^{**} \geq Z_n$. Moreover

$$-Z_n^{**1}L = (B+UZ_{n-1}^{**}D)L = -DL$$

and as before the inverse $Z_n^{**} = \sum (I - Z_n^{*-1})^j$ exists and

$$L = Z_n^{**}DL$$

Since $Z_n^{**} \geq Z_n$, $UZ_n^{**}D - UZ_nD \geq 0$. Combining this with $L \gg 0$ and UZ_nDL converging to UL gives $U(Z_n^{**}D - Z_nD)L$ converging to 0. This means that $UZ_n^{**}D$ and UZ_nD converge to the same limit. From this the limits of $-Z_n^{*-1}$ and $-Z_n^{-1}$ are the same; as are those of Z_n^{**} and Z_n .

Approximate Inverse

Some time ago, using an elimination procedure for the inverse in solving the matrix quadratic equation, difficulties were experienced with the code and an iterative inversion was substituted. The result was very rapid convergence on all problems. The iterative process for finding A^{-1} ,

$$A_n = (2I + A_{n-1}A)A_{n-1}$$

was used and only a single step was needed for adequate accuracy. This simple combination of two iterative calculations produces a single iterative calculation. Rather than treating it as a computer approximation to the inverse iteration which might converge, direct analysis can establish a much stronger result. The iteration

$$(10) \quad Z_n = (2I + Z_{n-1}(B + UZ_{n-1}D))Z_{n-1}$$

converges if $Z_0 = -B^{-1}$.

The justification is based on an induction to show that

$$Z_n \geq Z_{n-1} \geq 0. \quad \text{Using } Z_n(B + UZ_nD) \geq -I$$

$$\begin{aligned} Z_1 &= (2I + (-B^{-1})(B + U(-B^{-1})D))(-B^{-1}) \\ &= -B^{-1} + (-B^{-1})U(-B^{-1})D(-B^{-1}) \\ &\geq Z_0 \end{aligned}$$

The second inequality for Z_0 is

$$Z_0(B+UZ_0D) = (-B^{-1})(B+U(-B^{-1})D) = -I + (-B^{-1})U(-B^{-1})D \geq -I.$$

In both cases the inequality is true because $-B^{-1}$, U , and D are non-negative.

In the general situation

$$\begin{aligned} Z_{n+1} &= (2I + Z_n(B+UZ_nD))Z_n \\ &= Z_n + (I + Z_n(B+UZ_nD))Z_n. \end{aligned}$$

Since $Z_n(B+UZ_nD) \geq -I$, both $I + Z_n(B+UZ_nD) \geq 0$ and $Z_n \geq 0$ and thus

$Z_{n+1} \geq Z_n \geq 0$. Moreover since

$$Z_n \geq -B^{-1} = \sum (I+B)^k \geq I,$$

Z_n has no non-zero rows. Thus if

$$2I + Z_n(B+UZ_nD) \geq I$$

with strict inequality in any location Z_{n+1} has at least one entry strictly greater than an entry in Z_n . Assuming

$$Z_n(B+UZ_nD) \geq -I,$$

rearranging gives

$$I + Z_n(B+UZ_nD) \geq 0$$

which implies

$$(I + Z_n(B+UZ_nD))^2 \geq 0.$$

Expanding the square

$$I + 2Z_n(B+UZ_nD) + (Z_n(B+UZ_nD))^2 \geq 0$$

or

$$(11) \quad (Z_n(B+UZ_nD))^2 \geq -2Z_n(B+UZ_nD) - I.$$

Now

$$Z_{n+1}(B+UZ_{n+1}D) = (2I+Z_n(B+UZ_nD))Z_n \quad B + UZ_{n+1}D.$$

Using the non-negativity of the first bracket, U , D , Z_{n+1} and $Z_{n+1} \geq Z_n$

$$\begin{aligned} Z_{n+1}(B+UZ_{n+1}D) &\geq (2I+Z_n(B+UZ_nD))Z_n \quad B + UZ_nD \\ &= 2Z_n(B+UZ_nD) + (Z_n(B+UZ_nD))^2. \end{aligned}$$

Applying (11) gives

$$Z_{n+1}(B+UZ_{n+1}D) \geq 2Z_n(B+UZ_nD) - 2Z_n(B+UZ_nD) - I = -I$$

and the induction is complete.

Next again by induction establish $Z_n DL \leq L$. For Z_0 this is $(-B^{-1})DL \leq L$. This is necessary from the communication assumption that the probability of leaving a set of states must be 1 for all starting states or,

$$L = \sum_{n=0}^{\infty} (I+B)(U+D)L = -B^{-1}(UL+DL).$$

Since U , D , and Z_n are non negative,

$$Z_{n+1}D = ((2I + Z_n(B + UZ_nD))Z_nD) \leq 2Z_nD + Z_nBZ_nD.$$

From this assuming $Z_nDL \leq L$ and $Z_nUL \leq 0$,

$$Z_{n+1}DL \leq 2Z_nDL + Z_nBL = 2Z_nDL - Z_nUL - Z_nDL \leq Z_nDL \leq L$$

and the induction is complete.

Now $Z_nD \geq 0$ and $Z_nDL \leq L$ implies that the entries in Z_nD are bounded by 1. The bound and the monotonicity, $Z_n \geq Z_{n-1}$, imply that the sequence Z_nD converges. This implies that $B + UZ_nD$ converges. Since $Z_nDL \leq L$ and $Z_n \geq 0$, $I + B + UZ_nD \geq 0$ with $(I + B + UZ_nD)L \leq L - DL < L$

$$0 \leq \Sigma (I + B + UZ_nD)^k = -(B + UZ_nD)^{-1}.$$

This means that

$$Z_n(B + UZ_nD) \geq -I$$

implies

$$-Z_n \geq (B + UZ_nD)^{-1}$$

or

$$Z_n \leq -(B + UZ_nD)^{-1}.$$

Thus for n sufficiently large, Z_n is bounded by the limit on the right and the sequence Z_n converges.

For the limit Z_∞ ,

$$(2I + Z_\infty(B + UZ_\infty D)) \geq I.$$

If the inequality is strict in any coordinate so that

$$(2I + Z_{\infty}(B + UZ_{\infty}D)) = I + E$$

with $E \geq 0$,

$$(2I + Z_{\infty}(B + UZ_{\infty}D))Z_{\infty} = (I + E)Z_{\infty} = Z_{\infty} + EZ_{\infty}.$$

Since $Z_{\infty} \geq I$, $EZ_{\infty} \geq E$, which contradicts the requirement that $EZ_{\infty} = 0$.

Thus,

$$2I + Z_{\infty}(B + UZ_{\infty}D) = I$$

or

$$Z_{\infty}(B + UZ_{\infty}D) = -I$$

and

$$Z_{\infty} = -(B + UZ_{\infty}D)^{-1}.$$

Thus Z_{∞} has an inverse and

$$I + BZ_{\infty} + UZ_{\infty}DZ_{\infty} = 0$$

which has a unique solution $Z_{\infty} \geq 0$, $Z_{\infty}DL \leq I$. Thus this iteration also converges to the desired Z_{∞} .

As with the previous iterative schemes, $-B^{-1}$ is not the only and perhaps not the best choice of Z_0 . The required properties are

$$Z_0 \geq I$$

$$Z_0DL \leq I.$$

$$Z_0(B + UZ_0D) \geq -I.$$

Convergence of the Chain

The analysis of the quadratic Matrix equation may now be applied to the Markov chain. The Markov chain is ergodic if and only if there is a smallest $\Gamma < 1$ and a vector $P \gg 0$ satisfying

$$P(U + \Gamma B + \Gamma^2 D) = 0$$

Such a P and Γ imply the existence of $Q \geq 0$ with largest eigenvalue Γ satisfying

$$U + QB + Q^2 D = 0$$

A positive vector for eigenvalue 1 for the transition operator can be defined using Q . This eigenvector is obtained by using $P_{i+1} = P_i Q$ to construct a truncated Markov chain involving the sets of states from 0 to i . The transition operator is

$$\left(\begin{array}{ccc} I + B_0 & U_0 & 0 \\ D_1 & I + B_1 & U_1 \\ 0 & & I + B_{i-1} & U_{i-1} \\ & & D_i & I + B + QD_i \end{array} \right)$$

From the iterative calculation QD has a positive entry in location r,s if state i,r communicates with i,s through states in sets with higher index values than i . Thus the truncated process is acyclic because the original chain is. The fact that the truncated operator is a conservative transition matrix follows from the fact that $QD \geq 0$ and $QDL = UL$. This means that $(U + I + B + QD)L = L$ and the row sums of the last sub-matrix row of the truncated

process are 1's. Thus the matrix is a conservative transition matrix of a finite state acyclic chain and has a unique eigenvector for eigenvalue 1. The full eigenvector is completed by using $P_j = P_{j-1}Q$ for $j > i$. The resulting vector is determined only up to a scale which must be selected so that entries in the vector sum to 1. This may be done if and only if $\Gamma < 1$. The result is not only the eigenvector of eigenvalue 1 but also the limiting probabilities for the Markov chain. The limit exists whenever a positive eigenvector with coordinates which sum to 1 can be constructed. When $\Gamma \geq 1$ the same construction produces a vector which cannot be normalized to have a coordinate sum to 1. Moreover there is no other matrix which can generate an eigenvector with geometric upper tail. From previous results [4], if a limit exists for the Markov chain it has upper tail generalized geometric structure. Thus when $\Gamma \geq 1$ the chain is null.

Alternative Convergence Criterion

Further information about Γ and P can be found by considering the matrix

$$T(\Gamma) = I + U + \Gamma B + \Gamma^2 D$$

For all Γ in the interval $(0,1)$, $T(\Gamma) \geq 0$. The only possible question occurs for the diagonal entries, but $\Gamma B_{jj} \geq B_{jj}$ for $\Gamma > 0$ and $B_{jj} \leq 0$ and $1 + B_{jj} \geq 0$ since $I + B$ is a submatrix of a transition matrix. For $0 < \Gamma \leq 1$, $T(\Gamma)$ has non zero entries everywhere that $T(1)$ has. By assumption $T^k(1) \gg 0$ for some k , and this implies that $T^k(\Gamma) \gg 0$ for all Γ in $(0,1]$. Since $T^k(\Gamma)$ is strictly positive it has a largest positive eigenvalue $\gamma(\Gamma)$ and a strictly

positive left eigenvector P_Γ and strictly positive right eigenvector W_Γ .

Since $T(\Gamma)$ is analytic in Γ , the eigenvalue $\gamma(\Gamma)$ is analytic in Γ . Since $T(1)$ is assumed to have row sums equal to 1, $\gamma(1) = 1$. As Γ approaches 0, $T(\Gamma)$ approaches $I + U \geq I$ which implies that $\gamma(\Gamma) \geq 1$.

To examine $\gamma(\Gamma)$ between the extremes differentiate $P_\Gamma(T(\Gamma) - \gamma(\Gamma)I)W_\Gamma = 0$. Using $\dot{}$ to denote differentiation the result is

$$0 = P_\Gamma \dot{(T(\Gamma) - \gamma(\Gamma)I)} W_\Gamma + P_\Gamma (T'(\Gamma) - \gamma'(\Gamma)I) W_\Gamma + P_\Gamma (T(\Gamma) - \gamma(\Gamma)I) W'_\Gamma$$

Since $P_\Gamma [T(\Gamma) - \gamma(\Gamma)I] = [T(\Gamma) - \gamma(\Gamma)I] W_\Gamma = 0$,

$$0 = P_\Gamma (T'(\Gamma) - \gamma'(\Gamma)I) W_\Gamma = P_\Gamma (B + 2\Gamma D - \gamma'(\Gamma)I) W_\Gamma$$

At $\Gamma = 1$, W_Γ becomes the vector of 1's, L , and

$$\gamma'(1) P_1 L = P_1 (B + 2\Gamma D) L = P_1 (-UL + DL)$$

Since P_1 and L are strictly positive,

$$\gamma'(1) > 0 \text{ if } P_1 UL < P_1 DL$$

$$\gamma'(1) \leq 0 \text{ if } P_1 UL \geq P_1 DL$$

If $\gamma'(1) > 0$ there must be a $\Gamma^* < 1$ for which $\gamma(\Gamma^*) = 1$. This means that

$$P_{\Gamma^*} (U + \Gamma^* B + \Gamma^{*2} D) = 0$$

If $\gamma'(1) \leq 0$, if there is $\Gamma_1 < 1$, there is a second value $\Gamma_2 < 1$ for which $\gamma(\Gamma_2) = 1$.

If $\gamma(\Gamma_1) = \gamma(\Gamma_2) = 1$, $\Gamma_1 < \Gamma_2 < 1$, then

$$Q_0^{**} = W_{\Gamma_2} P_{\Gamma_2}$$

$$Q_n^{**} = (U + Q_{n-1}^{**2} D) (-B^{-1})$$

produces a bounded sequence of matrices. The previous arguments provide

$P_{\Gamma_2} Q_n^{**} = \Gamma_2 P_{\Gamma_2}$, $Q_{n,r,s}^{**} \leq \Gamma_2 \max P_{\Gamma_2,s} / P_{\Gamma_2,r} < \infty$ and $Q_n^{**} > Q_n$. From this sequence a convergent subsequence may be selected with limit Q^{**} with

$$P_{\Gamma_2} Q^{**} = \Gamma_2 P_{\Gamma_2}. \text{ Furthermore } Q_n^{**} \geq Q \quad Q_n^{**} DL \geq QDL = UL$$

$$-Q_n^{**} BL = (U + Q_{n-1}^{**2} D)L$$

$$Q_n^{**} (+UL + DL) = UL + Q_{n-1}^{**2} DL$$

$$\Gamma_2 P_{\Gamma_2} (+UL + DL) = P_{\Gamma_2} UL + \Gamma_2 P_{\Gamma_2} Q_{n-1}^{**2} DL$$

$$\Gamma_2 P_{\Gamma_2} UL + P_{\Gamma_2} (-UL + \Gamma_2 PDL) = \Gamma_2 P_{\Gamma_2} Q_{n-1}^{**2} DL$$

Since $P_{\Gamma_2} (U + \Gamma_2 B + \Gamma_2^2 D) = 0$, $(1 - \Gamma_2) P_{\Gamma_2} (UL - \Gamma_2 DL) = 0$

$$P_{\Gamma_2} UL = P_{\Gamma_2} Q_{n-1}^{**2} DL$$

For the limit

$$P_{\Gamma_2} UL = P_{\Gamma_2} Q^{**2} DL$$

If there were strict inequality in any coordinate $Q^{**2} DL \geq UL$ then there would be strict inequality for $P_{\Gamma_2} UL \geq P_{\Gamma_2} Q^{**2} DL$ and a contradiction.

Now begin the alternative iteration with

$$Z_0^* = -(B + Q^* D)^{-1}.$$

Since

$$Z_o^{*-1} L = -(B+Q^* D) L = DL$$

$$L = Z_o^* DL$$

Thus the sequence

$$Z_n^* = -(B+UZ_{n-1}^* D)^{-1}$$

converge to a limit. Moreover this limit has the property $P_{\Gamma_1} UZ_n^* = \Gamma_1 P$ assuming Γ_1 is the smallest Γ for which $\gamma(\Gamma) = 1$. On the other hand

$$P_{\Gamma_2} Z_o^{*-1} = P_{\Gamma_2} [-(B+Q^* D)]$$

$$P_{\Gamma_2} Z_o^{*-1} = \frac{1}{\Gamma_2} P_{\Gamma_2} U$$

$$\Gamma_2 P_{\Gamma_2} = P_{\Gamma_2} UZ_o^*$$

and generally

$$P_{\Gamma_2} Z_n^{*-1} = P_{\Gamma_2} [-B+Z_{n-1}^* D]$$

$$P_{\Gamma_2} Z_n^{*-1} = P_{\Gamma_2} B - \Gamma_2 P_{\Gamma_2} D = \frac{1}{\Gamma_2} P_{\Gamma_2} U$$

and

$$\Gamma_2 P_{\Gamma_2} = P_{\Gamma_2} UZ_n^{*-1}$$

This is a contradiction since the limit would have two strictly positive eigenvectors with positive eigenvalues which was earlier shown impossible.

Since there can only be one $\Gamma < 1$ for which $\gamma(\Gamma) = 1$, no such value can exist for $\gamma'(1) \leq 0$ or when $P_1^{UL} \geq P_1^{DL}$. Thus the Markov chain has a limit if and only if $P_1^{UL} < P_1^{DL}$.

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